Abstract: Julia sets are considered one of the most attractive fractals and have wide range of applications in science and engineering. In recent decades, Rani and Kumar [26], introduced the Julia sets in superior orbits with improved escape criterions for the cubic polynomials. Our goal in this paper is to study the Julia sets in Noor orbit for the cubic polynomials $z \rightarrow z^3 + mz + n$. It is interesting to see that few cubic Julia sets are akin to Christmas tree, Sikh Mythological Symbol Khanda and some other decorative pictures.

Keywords: Julia sets, Four-step feedback system, escape criterion, Cubic equation.

I. INTRODUCTION

With the introduction of fractal geometry, mathematics has presented some interesting complex objects to computer graphics. Interest in Julia sets and related mathematics began in the 1920’s with Gaston Julia [18, p. 122]. His extraordinary talents were recognized from an early age and although he excelled in every subject, he was most passionate about mathematics. Despite the lack of computing power available at the time, Herald Cramer gave the first approximate image of Julia set. Julia sets as a topic of current research with much interest being describing the intricate structure of Julia sets and in calculating their various characteristics such as fractal dimension [30]. Julia sets have been studied for quadratic [8, 9, 14, 18], cubic [5, 6, 9, 10, 11, 15] and also for higher degree polynomials, under Picard orbit, which is an example of one-step feedback process.

Julia sets have been studied under the effect of noises [1-4, 29]. In 2004, Rani introduced superior iterates (a two-step feedback process) in the study of fractal theory, jointly with Kumar, and created superior Julia sets [26]. Later on, in a series of papers Rani, jointly with other researchers, generated and analyzed superior Julia sets for quadratic [13, 24, 27], cubic [19], and $n^k$ degree polynomials. Negi, Rani and Mahanti simulated behavior of Julia sets using switching process [16]. Also, superior Julia sets have been studied under the effect of noises [22, 23]. For a complete literature review of superior fractals one may refer to Singh, Mishra and Sinkala [28].

Recently, Ashish et. al. [25] have generated new Julia and Mandelbrot sets in Noor orbit (an example of four-step feedback process) with improved escape criterions. The purpose of this paper is to visualize cubic Julia sets in Noor orbit for the polynomial $z \rightarrow z^3 + mz + n$, where $z$, $m$ and $n$ are complex quantities. This paper has been divided in to five sections. In section 2, Noor iterative procedure has been defined, which is the basis of our work. In section 3, escape criterions have been derived for cubic polynomials under Noor orbit. Several cubic Julia sets have been generated in section 4. Finally, the paper has been concluded in section 5.

II. PRELIMINARIES

In recent literature, there are mainly three types of feedback machines one-step [18], two-step [19], and three-step [7, 12]. Here, we study Julia sets for cubic polynomial in Noor orbit [17] which is an example of four-step feedback process. It is better understood by following figure:

Definition 1. (Picard Orbit). Let $X$ be a non-empty set and $f : X \rightarrow X$. For a point $x_0$ in $X$, the Picard orbit (generally called orbit of $f$) is the set of all iterates of a point $x_0$, that is:

$O(f, x_0) = \{ x_n : x_n = f x_{n-1}, n = 1, 2, \ldots \}$.

where the $O(f, x_0)$ of $f$ at the initial point $x_0$ is the sequence $\{f^n x_0\}$.
Definition 2. (Noor Orbit). Let us consider a sequence \( \{ z_n \} \) of iterates for initial point \( z_0 \in X \) such that,
\[
NO(T, z_0, a^n, \beta^n, \gamma^n) = \{ z_{n+1} = (1-a^n)z_n + a^n Tz_n; \quad y_n = (1-\beta^n)x_n + \beta^n Tz_n; \quad z_n = (1-\gamma^n)x_n + \gamma^n Tz_n; \quad n = 0, 1, 2, \ldots \}
\]
where \( a^n, \beta^n, \gamma^n \in [0, 1] \) and \( \{ a^n \}, \{ \beta^n \}, \{ \gamma^n \} \) are sequences of positive numbers, which is called as Noor–orbit (NO). The above sequence of iterates is called as Noor iterates [17].

Notice that at \( \gamma^n = 0 \), NO reduces to Ishikawa orbits; at \( \beta^n = \gamma^n = 0 \), NO reduces to Mann orbit; and at \( \beta^n = \gamma^n = 0 \) and \( a^n = 1 \), it behaves as Picard orbit. In our further section, we have chosen \( a^n = a, \beta^n = \beta \) and \( \gamma^n = \gamma \) to make the analysis simpler.

Definition 3. (Julia Sets) The filled in Julia set of the function \( A \) is defined as \( K(A) = \{ z \in C : A^k(z) \) does not tend to \( \infty \} \), where \( C \) is a complex space, \( A^k(z) \) is \( k \)’th iterate of function \( A \) and \( K(A) \) denotes the filled in Julia set. The Julia set of the function \( A \) is defined to be the boundary of \( K(A) \) i.e.
\[
J(A) = \partial K(A)
\]
where \( J(A) \) denotes the Julia set. The set of points whose orbits are bounded under the Picard orbit \( A_{\alpha,n}(z) = z^3 + mz + n \) is called the Julia set. We choose the initial point 0, as 0 is the only critical point of \( A_{\alpha,n} \) [9, p. 225].

III. ESCAPE CRITERIONS FOR CUBIC POLYNOMIALS IN NO

Escape criterion plays an important role in the construction of Julia sets of a function. In this section, we present the new escape criteria for the following cubic polynomials:
\[
A_{m,n}(z) = z^3 + mz + n,
\]
where \( m \) and \( n \) are complex numbers.

Theorem 1. Let \( |z| > |\eta| > |\eta + 2| |\eta|^{1/2}, |z| > |\eta| + 2 |\eta|^{1/2} \) and \( |z| > |\eta| + 2 |\eta|^{1/2} \) exists, where \( 0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1 \) and \( a \) and \( b \) are in complex plane. Define
\[
z_1 = (1-\alpha)z + \alpha A_{m,n}(z)
\]
\[
z_n = (1-\alpha)z_{n-1} + \alpha A_{m,n}(z_{n-1}), \quad n = 2, 3, 4, \ldots
\]
where \( A_{m,n}(z) \) is the function of \( \gamma \) then \( |z_n| \to \infty \) as \( n \to \infty \).

Corollary 1. (Escape Criterion). Let \( A_{m,n}(z) = z^3 + mz + n \), where \( m \) and \( n \) are complex numbers.
Suppose \( |z| > \max\{|\eta|, |\eta + 2| \alpha, |\eta + 2| \beta, |\eta + 2| \gamma\} \)
then \( |z_n| \to \infty \) as \( k \to \infty \). Which gives the escape criterion for cubic polynomials.

Corollary 2. Suppose \( |z| > \max\{|\eta|, |\eta + 2| \alpha, |\eta + 2| \beta, |\eta + 2| \gamma\} \)
then \( |z_n| \to \infty \) as \( k \to \infty \). This corollary shows the Julia set of \( A_{m,n}(z) \), for any \( m \) and \( n \).

IV. GRAPHICAL REPRESENTATION OF CUBIC JULIA SETS

Using the computational work in Mathematica, we generated few Julia sets for cubic polynomials. We iterate the cubic polynomial \( z \to z' + mz + n \) in the Noor iterative procedure. Further, it has been observed that there are two sets of the parameters, \( a, \beta, \gamma, m, \) and \( n \). We observed the following results by changing the parameteric values of these pairs:

- From the Fig 1-10, we noticed that on increasing the absolute values of two pairs simultaneously or individually, the domain of the cubic Julia sets becomes small and connectivity of the set gets to decrease.
- It can be observed from the Fig 1-4 that prisoner se of the Julia sets gets fattier and connectivity of Julia sets gets increase when the absolute values of the parameter in any one set is increasing and in another set is decreasing.
- Further, from the Fig 1-6, it has been visualized that the Fig 1 and Fig 2 took the shapes of Christmas tree (an annual commemoration of birth of Jesus Christ), the Fig 5 and Fig 6 looks like the Sikh Mythological Symbol “Khanda”, and the Fig 3 is like wall decorative picture.

Fig.1 Christmas tree Cubic Julia Set for \( a=0.3, \beta=0.1, m = 0.5 + 0.255, n = 51 \)
Fig. 2 Christmas tree Julia Set for $\alpha=0.3, \beta=\gamma=0.1, m = 0.5 + 0.255, n = 71$

Fig. 3 Cubic Julia Set for $\alpha=0.3, \beta=\gamma=0.1, m = 0.5 + 0.255, n = -71$

Fig. 4 Cubic Julia Set for $\alpha=0.3, \beta=\gamma=0.1, m = 81, n = 0.5 + 0.2551$

Fig. 5 Khanda Cubic Julia Set for $\alpha=0.5, \beta=\gamma=0.05, m = 2.5 + 0.11, n = 1.51$

Fig. 6 Khanda Cubic Julia Set for $\alpha=0.9, \beta=\gamma=0.1, m = 3, n = 0.5 + 0.2551$

Fig. 7 Cubic Julia Set for $\alpha=\beta=0.1, \gamma=0.5, m = 0, n = -3.551$
V. CONCLUSION

In the dynamics of cubic polynomial \( z \rightarrow z^3 + mz + n \) there exists many cubic Julia sets for the values of parameters \( \alpha, \beta, \gamma, m, n \) and \( z \) with respect to Noor orbit (NO). It is interesting to see that few Julia sets took the shapes of Christmas tree, Sikh Mythological Symbol “Khanda” and wall decorative picture.

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